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On high-frequency asymptotics in diffraction by finite-length waveguides: Open structures

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Abstract. The paper is concerned with high-frequency diffraction by open finite-length waveguides. The problem is reduced to an integral equation of the first kind over a finite interval, with its kernel depending on the difference of the arguments only. A new asymptotic approach is developed that permits explicit analytical representation of the solution.

Key words: diffraction, waveguide, integral equation, asymptotics.

1. Introduction

Wave propagation in waveguides is a very important practical problem in the fields of acoustics, dynamics of seismic waves, ultrasonic nondestructive testing of materials, antenna design and optical resonators [1–5]. From a theoretical point of view the case of a homogeneous infinite waveguide of constant width has been studied by means of the Fourier transform and the obtained results have now become classical (see [6]). Some relations between mode- and ray-properties have also been discovered. A powerful mathematical technique developed by N. Wiener, E. Hopf, and V. A. Fok [7] has been adapted for the case of semi-infinite waveguides. Through application of this method, many problems for open-ended waveguides have been successfully investigated [8]. Particularly, in the short-wave regime the technique permits to transfer the geometrical theory of diffraction (Keller [9]) from the case of thin slits to the semi-infinite waveguide (Lee [10]), with the use of factorization of the main characteristic function. It should be noted that the investigation of diffraction problems within the different great scientific schools, in underwater acoustics, solid dynamics, and electromagnetics, was often carried out independently and sometimes they duplicated each other. Nevertheless, the next intrinsic subject for all of them during the 50–70s had become the consideration of finite-length structures. It was clarified that the wave field in elongated waveguides is locally similar to that in infinite ones (Jones [11]). Later it was determined more exactly that these relations take place for low frequencies only. However, a special approach was developed for arbitrary fixed frequency which permits reduction to a finite linear algebraic system, to eliminate the difference between semi-infinite and elongated waveguides (Babeshko [12]).

Thus, the main analytical results for homogeneous waveguides of constant thickness were obtained at the beginning of 80s. Since then the rapid increase in computer power generated the widespread tendency to treat engineering problems numerically, and many diffraction problems have been solved with the use of finite-element and boundary-element techniques.

However, numerical methods have obvious disadvantages, since they can usually help to obtain only concrete digits for the values of the physical unknowns and do not allow one, as a rule, to extract qualitative properties of any physical process. Moreover, if the diffraction problem for a finite structure is studied numerically, then it is necessary to take at least 8–10 grid nodes for every wave-length. Thus, if we solve the short-wave scattering problem, the discretization procedure involves a discrete algebraic system which is too large, even for modern computers. To overcome this difficulty, this case should be considered with the use of a high-frequency asymptotic method.

From an engineering point of view, we aim to apply the results obtained below in the ultrasonic evaluation of materials. The incident ultrasonic wave produced by a transducer at the surface of the specimen, for the purpose of detecting defects in the material, is diffracted by these defects. Very often the flaws appear to be like thin slits or cracks – single or multiple. Wave-lengths in acoustic microscopy can get as small as 10^{-4} m, with the size of the flaw being of the order of 10^{-2} m. Thus, all geometric parameters are about a hundred times larger than the wave-length (in the optical range this can be about one thousand times). So, a high-frequency analysis is very suitable in this field. As to the single scatterer, an interesting asymptotic method was developed in the fifties (see for instance, Millar [13]), which is founded upon successive solution of the Wiener-Hopf equations for semi-infinite slits. This method, although being proposed first as the result of some physical ideas (the slit edges slightly influence each other), was later justified mathematically and is now known as the method of the ‘edge waves’. The simplest array of cracks in materials is a pair of equal cracks, and diffraction of the ultrasonic waves by the simple flaw structure involves the study of the wave processes in the finite-length waveguide. The physics of the wave process near the coupled structure is more complex, owing to interactions between the faces.

The ‘edge waves’ method cannot be transferred directly to the diffraction by a finite-length waveguide, due to the presence of the propagating mode waves that lead to a strong interaction between the ends of the waveguide. Nevertheless, for specific values of the frequency, close to these mode values, Vajnshtejn [14] first proposed, as the basis of some physical ideas, an approach similar to the method of ‘edge waves’. He discovered that at these frequencies the wave propagating along the waveguide does not allow any mode distinct from the incident one, when reflecting from the open edge. Hence, it is a transformation of a one-type wave, like in diffraction by the single slit, and so successive reflections from the edges correctly describe the wave picture. Nobody knows if this idea is correct for arbitrarily high frequency, different from these critical mode values.

The main purpose of the present work is to put these physical ideas on a strict mathematical basis. We propose an asymptotic method valid for high frequencies antipodal to those of Vajnshtejn. Thus, our results complement the results of Vajnshtejn and others. Here we study only open structures when resonances are absent, and symbolic functions of the main integral operators have no pole singularities on the real axis.

2. Problem formulation and reduction to integral equations

Let the plane incident acoustic wave ($k = \omega/c$ is the wave number)

$$\varphi^{\text{in}} = \mathrm{e}^{-ik(x\cos\vartheta + y\sin\vartheta)} \tag{1}$$

be scattered by a finite-length waveguide with the incident angle ϑ, as illustrated in Figure 1. The wave process is assumed to be harmonic with the following dependence on

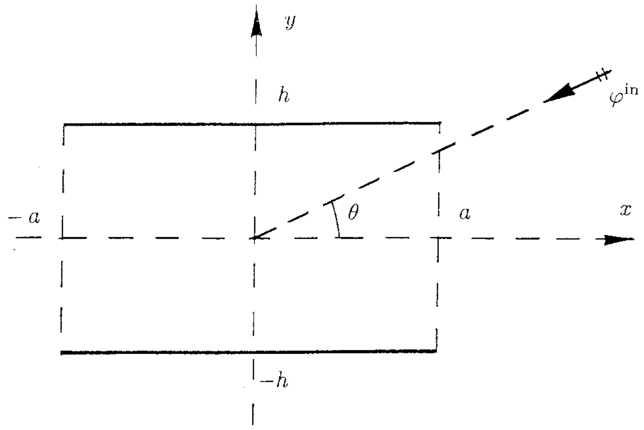

Figure 1. Incidence of a plane acoustic wave upon a finite-length waveguide.

time: $\exp(-i\omega t)$, where ω is the frequency of the oscillations. The scattered wave potential $\varphi(x, y)$ is the difference between the total φ^t and the incident φ^{in} waves

$$\varphi = \varphi^t - \varphi^{\text{in}} \tag{2}$$

and satisfies the Helmholtz equation

$$\Delta\varphi + k^2\varphi = 0, \tag{3}$$

everywhere outside the two finite parallel plates of length $2a$, with the distance between them being equal to $2h$. Assume, for instance, that the plates are acoustically soft

$$\varphi^t = 0 : \varphi = -\varphi^{\text{in}}, \quad |x| \leqslant a, \quad y = \pm h. \tag{4}$$

This problem was studied by D. S. Jones [11] who used a method which is unsuitable for very high frequencies. We consider here the particular problem as an example only, for the purpose of demonstrating the asymptotic technique which can be applied to the case of arbitrary incident wave and arbitrary open geometry of the waveguide, in the short-wave regime.

By applying the Fourier transform along the x-axis

$$\Phi(\alpha, y) = \int_{-\infty}^{\infty} \varphi(x, y)\, \mathrm{e}^{i\alpha x}\, \mathrm{d}x, \qquad \varphi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\alpha, y)\, \mathrm{e}^{-i\alpha x}\, \mathrm{d}\alpha \tag{5}$$

to the Helmholtz Equation (3) (see [5]), we obtain the following representation for the function $\Phi(\alpha, y)$

$$\Phi(\alpha, y) = \begin{cases} A(\alpha)\,\mathrm{e}^{-\beta y}, & y > h, \qquad \text{(6a)} \\ C(\alpha)\,\mathrm{e}^{-\beta y} + D(\alpha)\,\mathrm{e}^{\beta y}, & |y| \leqslant h, \qquad \text{(6b)} \\ B(\alpha)\,\mathrm{e}^{\beta y}, & y < -h, \qquad \text{(6c)} \end{cases}$$

where $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$ are unknown functions which should be defined from the boundary conditions and $\beta = \sqrt{\alpha^2 - k^2}$. In the definition of the branching function $\beta(\alpha)$ we use the branch cut such that $\operatorname{Re}\beta \geqslant 0$ for arbitrary complex-valued α.

It is evident that the boundary condition (4) provides continuity of the potential $\varphi(x, y)$ over the plates

$$\varphi|_{y=h+0} = \varphi|_{y=h-0} = -\varphi^{\text{in}}|_{y=h}; \qquad \varphi|_{y=-h+0} = \varphi|_{y=-h-0} = -\varphi^{\text{in}}|_{y=h}, \quad |x| \leqslant a. \tag{7}$$

Together with the continuity of the wave field outside the plates, this involves the equality

$$\varphi|_{y=h+0} = \varphi|_{y=h-0}; \qquad \varphi|_{y=-h+0} = \varphi|_{y=-h-0}, \quad -\infty < x < \infty \tag{8}$$

that gives the two relations

$$A(\alpha)\,\mathrm{e}^{-\beta h} = C(\alpha)\,\mathrm{e}^{-\beta h} + D(\alpha)\,\mathrm{e}^{\beta h}, \tag{9a}$$

$$B(\alpha)\,\mathrm{e}^{-\beta h} = C(\alpha)\,\mathrm{e}^{\beta h} + D(\alpha)\,\mathrm{e}^{-\beta h}. \tag{9b}$$

Let us introduce the new unknown functions $u(x)$ and $v(x)$ as follows

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=h+0} - \left.\frac{\partial\phi}{\partial y}\right|_{y=h-0} = \begin{cases} 0, & |x| > a \\ u(x), & |x| \leqslant a \end{cases}, \tag{10a}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=-h-0} - \left.\frac{\partial\phi}{\partial y}\right|_{y=-h+0} = \begin{cases} 0, & |x| > a \\ u(x), & |x| \leqslant a \end{cases}. \tag{10b}$$

The differences vanish here along $|x| > a$ because of the continuity of the wave field.

With the use of (5), the last equations are equivalent to

$$\beta[A(\alpha)\,\mathrm{e}^{-\beta h} + D(\alpha)\,\mathrm{e}^{\beta h} - C(\alpha)\,\mathrm{e}^{-\beta h}] = -\int_{-a}^{a} u(\xi)\,\mathrm{e}^{i\alpha\xi}\,\mathrm{d}\xi, \tag{11a}$$

$$\beta[B(\alpha)\,\mathrm{e}^{-\beta h} + C(\alpha)\,\mathrm{e}^{\beta h} - D(\alpha)\,\mathrm{e}^{-\beta h}] = \int_{-a}^{a} v(\xi)\,\mathrm{e}^{i\alpha\xi}\,\mathrm{d}\xi. \tag{11b}$$

It follows from (9), (11) that

$$C(\alpha) = \frac{\mathrm{e}^{-\beta h}}{2\beta}\int_{-a}^{a} v(\xi)\,\mathrm{e}^{i\alpha\xi}\,\mathrm{d}\xi; \qquad D(\alpha) = -\frac{\mathrm{e}^{-\beta h}}{2\beta}\int_{-a}^{a} u(\xi)\,\mathrm{e}^{i\alpha\xi}\,\mathrm{d}\xi, \tag{12}$$

with $A(\alpha)$ and $B(\alpha)$ being defined from (9). Lastly, the boundary condition (7) yields the following system of two integral equations

$$\frac{1}{2\pi}\int_{-a}^{a} u(\xi)\,\mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\alpha(\xi-x)}}{2\beta}\,\mathrm{d}\alpha - \frac{1}{2\pi}\int_{-a}^{a} v(\xi)\,\mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\alpha(\xi-x)-2\beta h}}{2\beta}\,\mathrm{d}\alpha$$
$$= \mathrm{e}^{-ik(h\sin\vartheta + x\cos\vartheta)}, \quad |x| \leqslant a, \tag{13}$$

$$\frac{1}{2\pi}\int_{-a}^{a} v(\xi)\,\mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\alpha(\xi-x)}}{2\beta}\,\mathrm{d}\alpha - \frac{1}{2\pi}\int_{-a}^{a} u(\xi)\,\mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\alpha(\xi-x)-2\beta h}}{2\beta}\,\mathrm{d}\alpha$$
$$= -\mathrm{e}^{ik(h\sin\vartheta - x\cos\vartheta)}, \quad |x| \leqslant a. \tag{14}$$

By applying summation and subtraction to (13) and (14), we can reduce the last system to a pair of two independent integral equations

$$\int_{-a}^{a} [u(\xi) + v(\xi)] K_1(x - \xi)\, \mathrm{d}\xi = -4i\, \sin(kh\, \sin\vartheta)\, \mathrm{e}^{-ikx\cos\vartheta}, \quad |x| \leqslant a, \tag{15}$$

$$\int_{-a}^{a} [u(\xi) - v(\xi)] K_2(x - \xi)\, \mathrm{d}\xi = 4\, \cos(kh\, \sin\vartheta)\, \mathrm{e}^{-ikx\cos\vartheta}, \quad |x| \leqslant a, \tag{16}$$

where

$$K_{1,2}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{1,2}(\alpha)\, \mathrm{e}^{-i\alpha x}\, \mathrm{d}\alpha, \tag{17}$$

$$G_1(\alpha) = \frac{1 - \mathrm{e}^{-2\beta h}}{\beta}; \qquad G_2(\alpha) = \frac{1 + \mathrm{e}^{-2\beta h}}{\beta}. \tag{18}$$

Note that the symbolic functions $G_1(\alpha)$ and $G_2(\alpha)$ have the branch points $\alpha = \pm k$, and there are no singularities, except these two points, which is an intrinsic property of open structures. The closed ones may possess a countable set of simple poles, with a finite number of them being distributed on the real axis. When any pole coincides with the origin, we have a resonant case (see [12]). As shown below, the distribution of the zeros of the symbol $G_{1,2}(\alpha)$ is important too. If a small attenuation is added to the medium, then the positive zeros and the branch point $\alpha = k$ move upwards, and the negative ones downwards [5]. This implies that for an ideal medium, for which these points are situated on the real axis, the integration contour σ in (17) should bend around the positive points from below and the negative ones from above.

It is conventionally recognized that a presence of branch points in the symbolic function complicates analytical treatment. The algorithm developed below is insensitive to the influence of the branching effect.

3. High-frequency asymptotic solution of the main integral equations

Obviously, in order to solve Equations (15) and (16), it is sufficient to treat the following equation written in a nondimensional form

$$\int_{-b}^{b} w(\xi) K(x - \xi)\, \mathrm{d}\xi = f(x); \quad |x| \leqslant b, \quad \chi = kh, \quad b = a/h, \tag{19a}$$

$$u(x) \pm v(x) = \left\{ \begin{array}{c} -4i\, \sin(\chi\, \sin\vartheta) \\ 4\, \cos(\chi\, \sin\vartheta) \end{array} \right\} \frac{w(x)}{h}; \qquad f(x) = \mathrm{e}^{-i\chi x\cos\vartheta}, \tag{19b}$$

$$K(x) = \frac{1}{2\pi} \int_{\sigma} G_{1,2}(\alpha)\, \mathrm{e}^{-i\alpha x}\, \mathrm{d}\alpha, \qquad G_{1,2}(\alpha) = \frac{1 \mp \mathrm{e}^{-2\gamma}}{\gamma}; \qquad \gamma = \sqrt{\alpha^2 - \chi^2}. \tag{19c}$$

There are two independent dimensionless parameters in this problem – the frequency parameter χ and the relative length of the waveguide $b = a/h$. The asymptotic analysis undertaken by Jones [11] implies a so-called one-mode regime ($\pi < \chi < 2\pi$) with $b \to \infty$. We study the opposite case $\chi = kh \to \infty$ with bounded b (so $ka \to \infty$ too), which is more complex, since the number of propagating modes grows with increasing frequency.

Let us represent (following Aleksandrov [13]) the solution of Equation (19) as a superposition of the three functions

$$w(x) = w_1(b+x) + w_2(b-x) - w_0(x), \tag{20}$$

with the new ones satisfying the following system equivalent to the initial Equation (19)

$$\begin{aligned}\int_{-b}^{\infty} & w_1(b+\xi)K(x-\xi)\,\mathrm{d}\xi \\ &= f(x) + \int_{-\infty}^{-b}[w_2(b-\xi) - w_0(\xi)]K(x-\xi)\,\mathrm{d}\xi, \quad (-b < x < \infty),\end{aligned} \tag{21a}$$

$$\begin{aligned}\int_{-\infty}^{b} & w_2(b-\xi)K(x-\xi)\,\mathrm{d}\xi \\ &= f(x) + \int_{b}^{\infty}[w_1(b+\xi) - w_0(\xi)]K(x-\xi)\,\mathrm{d}\xi, \quad (-\infty < x < b),\end{aligned} \tag{21b}$$

$$\int_{-\infty}^{\infty} w_0(\xi)K(x-\xi)\,\mathrm{d}\xi = f(x), \quad (-\infty < x < \infty). \tag{21c}$$

From a physical point of view the integral operators on the left-hand sides of (21) are related to one infinite waveguide and two semi-infinite ones. For a wide class of physical problems additional integrals on the right (21a, b), the so-called ‘tails’, appear to be small in some sense. For instance, if the frequency is low enough, so that there is no real zero of the symbol $G_{1,2}(\alpha)$ (k is less than the first critical value), then for a long waveguide ($b = a/h \gg 1$ – the asymptotic case considered by Jones [11]) the differences in the square brackets vanish when $\xi \to \pm\infty$. Thus, it can be proved that the integrals on the right tend to zero with $a \to \infty$. Physically it means that the edges of the thin open-ended waveguide do not influence each other in an asymptotic sense. A similar approach is possible for the short-wave diffraction problem for a single isolated plate (as well as for a single slot in the infinite plate). The symbolic function $G(\alpha)$ there may be formally obtained from (18) at $h \to \infty$: $G(\alpha) \sim 1/\gamma$, thus there are no zeros of the function $G(\alpha)$. It involves again an asymptotic disappearance of the right-hand integrals (21a, b), with the same physical meaning – the edges of the plate influence each other weakly when $a/\lambda \to \infty$ ($\lambda = 2\pi/k$ is the wave-length). The described approach generates the well known ‘edge-waves’ method (see [9]) which was developed by means of a different mathematical technique.

In the problem at hand the zeros $\pm\alpha_{1m}$, $\pm\alpha_{2m}$ of the function $G_{1,2}(\alpha)$

$$\alpha_{1m} = \sqrt{\chi^2 - (\pi m)^2}; \qquad \alpha_{2m} = \sqrt{\chi^2 - (\pi m - \tfrac{1}{2}\pi)^2}, \quad m = 1, 2, \ldots \tag{22}$$

play the most important role.

The number of positive zeros (22) increases with the growth of the frequency parameter $\chi = kh$. It generates more and more propagating ‘mode waves’ inside the waveguide (see [5]). Thus, the ends of the open finite-length waveguide just affect each other by these propagating waves, and the internal wave process differs considerably from that in semi-infinite structures. As a result, the differences $[w_2(b-\xi)-w_0(\xi)]$, $[w_1(b+\xi)-w_0(\xi)]$ do not vanish as $\chi \to \infty$.

In spite of the finite-length structure cannot be represented as a composition of two semi-infinite ones, there is a classical idea that diffraction by the open end of the finite-length waveguide is similar to diffraction by the edge of a semi-infinite one, when the wave number approaches any mode value. We mean the optical range of electromagnetics, where the both geometric sizes of the structure (a and h) are several thousand times larger than the wave length (see [14]). Below we determine in which sense this point of view is correct.

To begin with, let us write out the explicit representation for the kernels (see [8])

$$K_{1,2}(x) = \tfrac{1}{2} i\{H_0^{(1)}(\chi|x-\xi|) \mp H_0^{(1)}[\chi\sqrt{4+(x-\xi)^2}]\}, \tag{23}$$

where $H_0^{(1)}$ is the Hankel function of the first order with the following asymptotic behaviour

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, \mathrm{e}^{i(z-\pi/4)}; \quad z \to \infty. \tag{24}$$

It is clear from (23) and (24) that at high frequencies ($\chi \gg 1$) the kernel $K(x)$ in the right-hand sides of (21a, b) is strongly oscillating. So the right-hand integrals in (21a, b) may be small, if the differences in square brackets are asymptotically bounded.

Let us assume, as a first step, that these integrals can be neglected (correctness of this assumption should be verified afterwards). Then we can construct the solution of the Equations (21a, b) over the semi-infinite intervals using the Wiener-Hopf method. The derivation procedure is rather standard and routine (see for instance [5]). It gives the following result

$$\begin{aligned} w_1(x) \;=\; & \frac{\mathrm{e}^{-i\chi(x-b)\cos\vartheta}}{G(\chi\cos\vartheta)} - \frac{1}{2G_+(-\chi\cos\vartheta)}\sum_{m=1}^{\infty}\frac{(\pi m)^2}{\alpha_m(\alpha_m+\chi\cos\vartheta)} \\ & \times G_+(\alpha_m)\,\mathrm{e}^{i(\alpha_m x+\chi b\cos\vartheta)}, \end{aligned} \tag{25a}$$

$$\begin{aligned} w_2(x) \;=\; & \frac{\mathrm{e}^{i\chi(x-b)\cos\vartheta}}{G(\chi\cos\vartheta)} - \frac{1}{2G_+(\chi\cos\vartheta)}\sum_{m=1}^{\infty}\frac{(\pi m)^2}{\alpha_m(\alpha_m-\chi\cos\vartheta)} \\ & \times G_+(\alpha_m)\,\mathrm{e}^{i(\alpha_m x-\chi b\cos\vartheta)}, \end{aligned} \tag{25b}$$

where $G(\alpha) = G_+(\alpha)G_-(\alpha)$ is a factorization of the function $G(\alpha)$.

The solution of Equation (21c) is obtained by application of the Fourier transform as follows

$$w_0(x) = \frac{\mathrm{e}^{-i\chi x\cos\vartheta}}{G(\chi\cos\vartheta)}, \tag{25c}$$

so we can give the final representation for $w(x)$, with the accepted assumption, as

$$\begin{aligned} w(x) \;=\; & \frac{\mathrm{e}^{-i\chi x\cos\vartheta}}{G(\chi\cos\vartheta)} - \frac{1}{2}\sum_{m=1}^{\infty}(\pi m)^2\frac{G_+(\alpha_m)}{\alpha_m} \\ & \times\left\{\frac{\mathrm{e}^{i[\alpha_m(b-x)-\chi b\cos\vartheta]}}{(\alpha_m-\chi\cos\vartheta)G_+(\chi\cos\vartheta)} + \frac{\mathrm{e}^{i[\alpha_m(b+x)+\chi b\cos\vartheta]}}{(\alpha_m+\chi\cos\vartheta)G_+(-\chi\cos\vartheta)}\right\} \\ & +O(\chi^{-1/2}). \end{aligned} \tag{26}$$

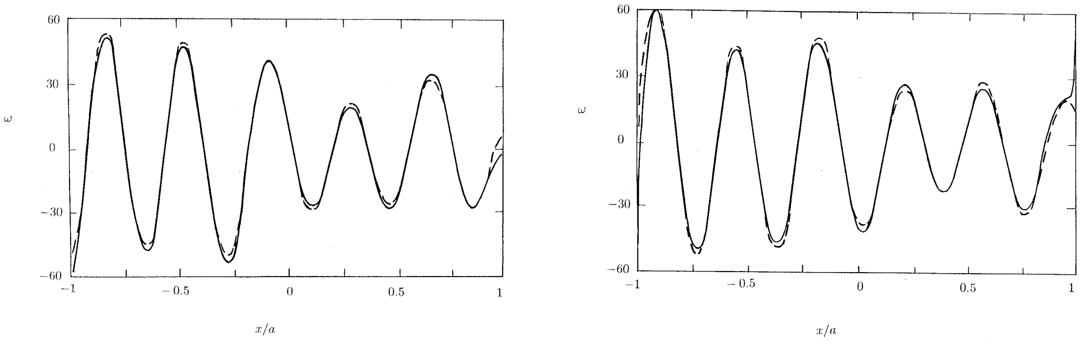

Figure 2. Comparison between exact (–) and asymptotic (- - -) solution of the integral Equation (19a), when $\theta = \pi/3$; $\chi = kh = 10{\cdot}5\pi$; $b = a/h = 1$; $G_1(\alpha) = [1 - \exp(-2\gamma)]/\gamma$: (a) $\mathrm{Re}[w(x)]$; (b) $\mathrm{Im}[w(x)]$.

The asymptotic estimate of the error of this formula is proved below.

The structure of the obtained solution (26) is the following. The first term corresponds to the case of an infinite waveguide and has the order $O(\chi)$, $\chi \to \infty$ (so far as $G(\chi \cos \vartheta) = i(1 \mp \exp(2\chi i \sin \vartheta))/(\chi \sin \vartheta))$. Several oscillating terms under the sum sign may appear to have the same order $O(\chi)$ wih respect to parameter χ (if $\alpha_m \sim \chi \cos \vartheta$). Thus, the behaviour of the solution is also highly oscillating.

When estimating the value of the right-hand integrals in (21a, b), let us note that the argument $(x - \xi)$ in the kernel $K(x - \xi)$ is of a constant sign for all x and ξ there, so $K(x - \xi) \sim O(1/\sqrt{\chi})$ uniformly on x and ξ. Thus, the integrand is of the order $O(\chi^{1/2})$, $\chi \to \infty$. Besides, there is no stationary point, which is directly opposite to the left-hand kernel with its stationary point $x - \xi = 0 \sim \xi = x$. Therefore, it can be shown, with the use of integration by parts that, owing to the oscillating structure of the integrand, the value of the integrals has the order $O(\chi^{1/2})/\chi = O(\chi^{-1/2})$, $\chi \to \infty$. This proves the correctness of the method when the right-hand integrals in (21a, b) are neglected. The only breakdown takes place when the value of the variable x coincides with the edge point ($x = -b$ in (21a) and $x = b$ in (21b)). For these values the argument $(x - \xi)$ may be small, and the asymptotic estimate (24) for the kernel is not valid. Thus, the developed approach gives the correct representation (26) all over the interval $-b < x < b$, excluding the ‘boundary layers’ where $b \pm x \sim 1/\chi$. This property can be clearly seen from Figure 2. Another restriction is related to the cases when the incident angle ϑ approaches those values where the incident wave generates standing modes. In these cases $G(\chi \cos \theta)$ tends to zero, $w_1(x)$ and $w_2(x)$ grow to infinity, and the right-hand integrals in (21a, b) become unbounded. Thus, for some particular values of the frequency and the angle of incidence the main asymptotic result (26) fails.

The main obtained formula (26) is worthy of special discussion. First of all, the fact that the wave amplitude grows along the waveguide faces with increasing frequency $(O(\chi), \chi \to \infty)$ is not described in the literature. Another interesting fact is that the mode waves do not decay with distance; however, some integrals of these wave functions decay, so they could be neglected in (21a, b). The nondecaying structure of the wave field in (26) shows that the solution has no relation to the problem with the semi-infinite waveguide where only one type of edge-wave, instead of two, is present.

It should also be noted that the asymptotic contribution to the sum in (26) is given only by nondecaying terms in the sum (when α_m is real). Therefore, in practice the summation can be restricted by $m = n$, if the frequency parameter is $\chi = \pi(n + \delta)$ ($0 < \delta < 1/2$, or $1/2 < \delta < 1$), where n is a large positive integer. Thus, the formula (26) is quite appropriate to evaluate the split function $w(x)$ at high frequencies, because efficient high-frequency representations for the coefficients $G_+(\alpha_m)$ are proposed in [5].

As soon as the solution of the main integral equation has been determined by (26), the scattered wave field $\varphi(x, y)$ can be obtained directly from (5), (6), and (12)

$$\begin{aligned}\varphi(x, y) &= \frac{i}{4}\int_{-a}^{a} v(\xi) H_0^{(1)}[k\sqrt{(x-\xi)^2 + (y+h)^2}]\,\mathrm{d}\xi \\ &\quad - \frac{i}{4}\int_{-a}^{a} u(\xi) H_0^{(1)}[k\sqrt{(x-\xi)^2 + (y-h)^2}]\,\mathrm{d}\xi, \qquad (27)\end{aligned}$$

uniformly over (x, y). The functions $u(x)$ and $v(x)$ here can be evidently expressed from (19b) in terms of the function $w(x)$, given by Equation (26) for the cases $G = G_1$ and $G = G_2$.

To assess the accuracy of the asymptotic formula (26) we applied a numerical method to solve the integral Equation (19). The main obstacle here is connected with the requirement that the numerical algorithm must provide stable calculations for arbitrarily high frequencies. Equation (19) is a Fredholm integral equation of the first kind, with the kernel (23) having a logarithmic singularity at $|x - \xi| \to 0$, like in classical boundary-element techniques. This is why we used a collocation technique which provides stability with a small mesh width ε, if the integration of the logarithmic part of the kernel is carried out explicitly.

To estimate the total number of nodes N, we note that we should take approximately 10 points over the wave-length, thus $N \sim 10ak/\pi$. Obviously, with increasing frequency parameter χ, the number of nodes N increases considerably. Therefore, the significance of the obtained asymptotic results becomes self-evident again.

An example of the comparison between the obtained analytical solution and the solution constructed by a direct numerical method is illustrated in Figure 2.

4. Conclusions

In this paper we have developed a self-consistent high-frequency asymptotic approach to the diffraction problem for open finite-length waveguides. The proposed method is characterized by the following features:

(1) It is well known (see [5]) that the presence of the branch points $\alpha = \pm\chi$ complicates the analytical analysis of the diffraction problems considerably. The proposed method is quite suitable for kernels containing branch points, as well as for kernels with meromorphic symbolic functions free of any branching.
(2) Generally, the wave field inside the finite-length structure cannot be represented as a composition of the wave fields in two semi-infinite waveguides. However, an asymptotic solution of the main integral equation can be constructed by means of a superposition of solutions for semi-infinite and infinite structures. This becomes possible thanks to certain asymptotic properties of the 'tails' in the integrals.
(3) The obtained results fail near the edges of the plates, as well as near the frequencies when the incident wave can generate standing modes. Thus, our results complement the classical results of Vajnshtejn developed only for frequencies close to the critical-mode values.

(4) It has been said (in some private discussions) that it is impossible to derive any explicit high-frequency asymptotics for the finite-length waveguide. However, the proposed method allows one to construct an explicit high-frequency representation which needs an additional numerical treatment to calculate some coefficients only.
(5) The main result of the work is given by the explicit formula (26) which takes two lines only. The upper limit of the summation in this formula may be chosen as $m = n$ (n is a number of positive zeros of $G_{1,2}(\alpha)$).
(6) The considered multi-mode case cannot be studied by a high-frequency asymptotic method applicable to a single slit, since the latter is free of wave modes. An asymptotic approach suitable for elongated waveguides with a fixed number of mode waves cannot be also applied to the considered problem, since the number of modes grows with increasing frequency.
(7) With the growth of the frequency the amplitude of the solution of the main integral equation (and the entire wave field as well) increases as $O(\chi)$. The structure of the solution becomes highly oscillatory as $\chi \to \infty$.
(8) The error of the obtained asymptotics is of the order O as $(\chi^{1/2})$, $\chi = kh \to \infty$. It is valid for arbitrarily fixed relative elongation $b = a/h$. Thus, the error slowly decreases with the frequency increasing; however Figure 2 demonstrates that the accuracy of the results is very good for moderate values of the frequency parameter – not ony for extremely high ones (in practice, from $\chi \geqslant 10$).

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